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Unicity of KAM tori

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Abstract. The classical KAM theorem establishes the persistence of invariant Lagrangean tori in nearly integrable Hamiltonian systems. These tori are quasi-periodic with Diophantine frequency vectors and their union is a nowhere dense set of positive measure in phase space. A long-standing question is to what extent the perturbed tori are unique. Using the fact that at the level of tori there exists a Whitney smooth conjugacy between the integrable approximation and its perturbation, we are able to prove this unicity. The unicity result is valid on a closed subset of the Diophantine torus union of full measure.

1. Introduction

Classical Kolmogorov–Arnold–Moser (KAM) theory deals with Hamiltonian perturbations of an integrable Hamiltonian system and proves the persistence of quasi-periodic (Diophantine) invariant Lagrangean tori. In [17] Pöschel proved the existence of Whitney smooth angle-action variables on a nowhere dense union of tori having positive Lebesgue measure (see also [8, 15]). This version of the KAM theorem can be formulated as a kind of structural stability restricted to a union of quasi-periodic tori. As such it is referred to as *quasi-periodic stability* [5, 6]. In this context, the conjugacy between the integrable system and its perturbation is smooth in the sense of Whitney. In this paper we treat only the case of Hamiltonian systems with Lagrangean invariant tori. Throughout we shall assume our systems to be smooth, in the sense of C^∞ . It should be noted that in the case of finite differentiability similar results hold, which are slightly weaker in the sense that certain well-controlled losses of differentiability occur.

Our goal is to investigate the unicity of the perturbed Diophantine tori in the following sense. First we delete a subset of the Diophantine frequency vectors of measure zero. Then, for the remaining frequencies we show that the Whitney smooth conjugacy between the Diophantine torus union in the integrable case and its perturbation is unique up to torus translation.

1.1. *Motivation.* The unicity of KAM tori has been a long-standing question, to which we propose a solution that uses the Whitney differentiability of the perturbed tori with respect to their frequencies [17–19]. This research was motivated by the attempt to prove a global KAM theory [2] for Lagrangean torus bundles in perturbations of (locally) integrable Hamiltonian systems that are non-trivial [10, 11]. The global KAM theorem keeps track of the bundle structure in the union of perturbed Diophantine tori, using Whitney differentiability. In this approach, the Lagrangean torus bundle is locally trivialized by angle-action charts and the question arises to what extent perturbed tori coincide in overlapping chart domains. The present paper ensures that this is indeed the case. We conjecture that the unicity result can be generalized to various other settings in KAM theory [5, 6], including cases of diffeomorphisms (such as area-preserving annulus maps).

1.2. *Formulation of the KAM theorem.* We start with a formulation of the KAM theorem [17] suited to our purposes [5, 6]. Therefore, for $n \geq 2$ we consider $A \subseteq \mathbb{R}^n$ as an open, bounded and connected subset. Also we let $\mathbb{T}^n = \mathbb{R}^n / (2\pi\mathbb{Z}^n)$ be the standard n -torus. The product $M = \mathbb{T}^n \times A$ is endowed with coordinates $(\alpha, a) = (\alpha_1, \alpha_2, \dots, \alpha_n, a_1, a_2, \dots, a_n)$, where the α_j are counted modulo $2\pi\mathbb{Z}$, and with a symplectic form $\sigma = d\alpha \wedge da = \sum_{j=1}^n d\alpha_j \wedge da_j$. Now take a smooth Hamiltonian function $H : M \rightarrow \mathbb{R}$, which is integrable in the sense that H does not depend on the angle variable α . The corresponding Hamiltonian vector field X_H , defined by $X_H \lrcorner \sigma = dH$, then takes the form

$$X_H(\alpha, a) = \omega(a) \frac{\partial}{\partial \alpha} = \sum_{j=1}^n \omega_j(a) \frac{\partial}{\partial \alpha_j}, \quad (1)$$

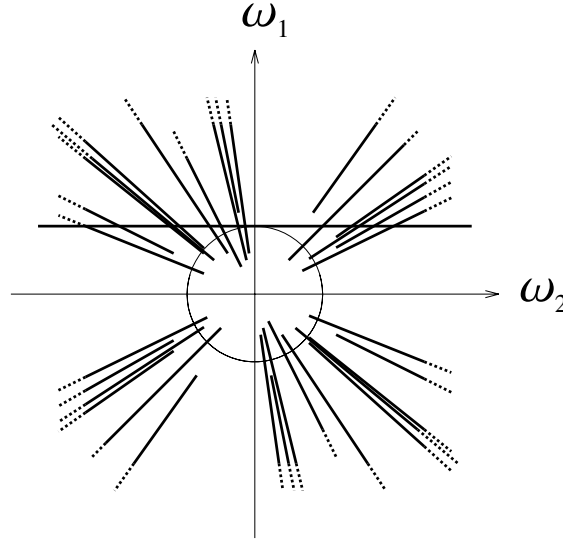
where $\omega(a) = \partial H / \partial a(a)$ is the frequency vector. We call $\omega : A \rightarrow \mathbb{R}^n$ the frequency map. We say that H is Kolmogorov non-degenerate if this frequency map is a diffeomorphism onto its image. Note that (1) exactly is the format of a Liouville integrable Hamiltonian system in angle-action variables (α, a) [1, 10].

Remark. Usually the above setting corresponds to a local trivialization of a bundle of Lagrangean invariant n -tori [2, 10, 11] which does not have to be globally trivial.

The KAM theorem deals with the persistence of certain invariant n -tori under small non-integrable perturbation. The tori of interest have Diophantine frequencies, to be defined next. Let $\tau > n - 1$ and $\gamma > 0$ be given. We define the set of (τ, γ) -Diophantine frequency vectors by

$$D_{\tau, \gamma}(\mathbb{R}^n) = \{\omega \in \mathbb{R}^n \mid |\langle \omega, k \rangle| \geq \gamma |k|^{-\tau} \text{ for all } k \in \mathbb{Z}^n \setminus \{0\}\}. \quad (2)$$

One easily sees that $D_{\tau, \gamma}(\mathbb{R}^n)$ is a closed subset with the following closed half-line property: whenever $\omega \in D_{\tau, \gamma}(\mathbb{R}^n)$ and $s \geq 1$, then also the scalar product $s \cdot \omega \in D_{\tau, \gamma}(\mathbb{R}^n)$; compare with Figure 1. The intersection $S_{\tau, \gamma} := D_{\tau, \gamma}(\mathbb{R}^n) \cap \mathbb{S}^{n-1}$ with the unit sphere again is a closed (even a compact) set. Application of the Cantor–Bendixson theorem [13] to this latter set yields that $S_{\tau, \gamma}$ is the union of a perfect and a countable set. Since the resonant hyperplanes (with equations $\langle \omega, k \rangle = 0, k \in \mathbb{Z}^n \setminus \{0\}$) give a dense web in the complement $\mathbb{S}^{n-1} \setminus S_{\tau, \gamma}$, it follows that this perfect set is totally disconnected.

FIGURE 1. Sketch of the set $D_{\tau, \gamma}(\mathbb{R}^2)$.

Summing up we conclude that the perfect subset of $\mathbb{S}_{\tau, \gamma}$ is a Cantor set. Moreover, the measure of $\mathbb{S}^{n-1} \setminus \mathbb{S}_{\tau, \gamma}$ is of order $O(\gamma)$ as $\gamma \downarrow 0$.

Consider the set $\Gamma = \omega(A)$ as well as the shrunk version $\Gamma_\gamma \subseteq \Gamma$ given by

$$\Gamma_\gamma = \{\omega \in \Gamma \mid \text{dist}(\omega, \partial\Gamma) > \gamma\}.$$

Let $D_{\tau, \gamma}(\Gamma_\gamma) = \Gamma_\gamma \cap D_{\tau, \gamma}(\mathbb{R}^n)$. From now on we will take γ sufficiently small to ensure that $D_{\tau, \gamma}(\Gamma_\gamma)$ is a nowhere dense set of positive measure. Finally, define the shrunk domain $A_\gamma = \omega^{-1}(\Gamma_\gamma) \subseteq A$ as well as its nowhere dense counterpart $D_{\tau, \gamma}(A_\gamma) = \omega^{-1}(D_{\tau, \gamma}(\Gamma_\gamma)) \subseteq A_\gamma$, where the measure of $A \setminus D_{\tau, \gamma}(A_\gamma)$ is of order $O(\gamma)$ as $\gamma \downarrow 0$.

Staying within the class of Hamiltonian systems, we perturb the Hamiltonian vector field $X = X_H$ in the C^∞ -topology, assuming that all vector fields have a C^∞ -extension to the closure $\mathbb{T}^n \times \bar{A}$. The C^∞ -topology is then generated by all C^m -norms $\|\cdot\|_m$ on $\mathbb{T}^n \times \bar{A}$, see [14].

THEOREM 1. (KAM [6, 17]) *Suppose that the integrable Hamiltonian system $X = X_H$ is Kolmogorov non-degenerate as a C^∞ -system on $\mathbb{T}^n \times A$. Then, for sufficiently small $\gamma > 0$ and all sufficiently C^∞ -small Hamiltonian perturbations \tilde{X} of X there exists a map $\Phi : \mathbb{T}^n \times A \rightarrow \mathbb{T}^n \times A$ with the following properties.*

- (1) Φ is a C^∞ -diffeomorphism onto its image, where, in the C^∞ -topology, whenever $\tilde{X} \rightarrow X$ also $\Phi \rightarrow \text{Id}$.
- (2) The map $\hat{\Phi} = \Phi|_{\mathbb{T}^n \times D_{\tau, \gamma}(A_\gamma)}$ conjugates X to \tilde{X} , that is, $\hat{\Phi}_* X = \tilde{X}$.

Remarks.

- (1) The map Φ , which maps $\mathbb{T}^n \times A_\gamma$ into $\mathbb{T}^n \times A$, is, in general, not symplectic. Notice that Theorem 1 asserts that the integrable system is (locally) quasi-periodically stable.

- (2) There are direct generalizations of Theorem 1 to the world of C^k -systems endowed with the C^k -topology [14] for k sufficiently large. For C^k -versions of the classical KAM theorem, see [5, 6, 17].

2. Unicity of KAM tori

In this section we show that the KAM tori as obtained in Theorem 1 are essentially unique. Using Kolmogorov non-degeneracy we may assume that our integrable Hamiltonian system (1) has the form

$$X(\alpha, a) = \sum_{j=1}^n a_j \frac{\partial}{\partial \alpha_j}. \quad (3)$$

Notice that the symplectic form is now no longer standard.

The KAM Theorem 1 asserts that, for any sufficiently small Hamiltonian perturbation \tilde{X} of X , there exists a near-identity diffeomorphism $\Phi : \mathbb{T}^n \times A \rightarrow \mathbb{T}^n \times A$, such that for any (τ, γ) -Diophantine frequency vector ω the restriction $\Phi|_{\mathbb{T}^n \times \{\omega\}}$ conjugates $X|_{\mathbb{T}^n \times \{\omega\}}$ to an \tilde{X} -invariant n -torus. The problem addressed in this paper is the unicity of the diffeomorphism Φ . Outside the Diophantine tori it is not unique at all, but for most $\omega \in D_{\tau, \gamma}$ it turns out to be unique up to torus translations. A different formulation of this problem is as follows. Consider the pull back vector field $\tilde{X} = \Phi_*^{-1}(\tilde{X})$. Note that \tilde{X} coincides with X on the (τ, γ) -Diophantine tori and is C^∞ -close to X . We aim to show that for most $\omega \in D_{\tau, \gamma}$ any other \tilde{X} invariant n -torus conjugate to $X|_{\mathbb{T}^n \times \{\omega\}}$ has to be ‘far away’ from $\mathbb{T}^n \times \{\omega\}$, in the sense that the corresponding conjugacy is far from the identity map.

Remark. Consider the constant vector field $\mathbb{X}_\omega = \sum_{j=1}^n \omega_j (\partial/\partial \alpha_j)$, on the standard torus $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z}^n)$ with non-resonant frequency vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)$. The self-conjugacies of \mathbb{X}_ω are exactly the translations of \mathbb{T}^n . This directly follows from the fact that each trajectory of \mathbb{X}_ω is dense. Note that these translations determine the affine structure on \mathbb{T}^n . For an arbitrary vector field X with an invariant n -torus T , we may define quasi-periodicity of $X|_T$ by requiring the existence of a smooth conjugacy $\phi : T \rightarrow \mathbb{T}^n$ with a vector field \mathbb{X}_ω on \mathbb{T}^n , i.e. such that $\phi_*(X|_T) = \mathbb{X}_\omega$. In that case, the self-conjugacies of $X|_T$ determine a natural affine structure on T . Note that the translations on T and on \mathbb{T}^n by ϕ are conjugate and that therefore the conjugacy ϕ itself is unique modulo torus translations.

Finally, observe that, in the present integrable Hamiltonian case, the (local) angle-action variables (α, a) give rise to X -invariant tori T_a . Also the involutive integrals give rise to an affine structure [1, 10]. We note that on the quasi-periodic tori T_a the latter structure exactly coincides with the one we introduced above. Indeed, in the angle coordinates α the vector field X becomes constant. For details see [2].

2.1. C^∞ -extension and density points. First we need to introduce an appropriate subset of $D_{\tau, \gamma}(\mathbb{R}^n)$. In general, let $K \subseteq \mathbb{R}^n$ be a closed set. We say that $a \in K$ is a *density point* of K precisely if any C^∞ map $F : \mathbb{R}^n \rightarrow \mathbb{R}$, such that $F|_K = 0$, has an infinite-jet $j^\infty(F)(a) = 0$. The set of all density points of K is denoted by K^* . Moreover, in general

we say that the closed set $K \subseteq \mathbb{R}^n$ has the *closed half-line property* if the following holds: whenever $p \in K$ and $s \geq 1$, then also $sp \in K$.

LEMMA 1. (Properties of K^*) *Let $K \subseteq \mathbb{R}^n$ be a closed set. Then:*

- (1) $K^* \subseteq K$ is a closed set;
- (2) $K \setminus K^*$ has Lebesgue measure zero;
- (3) if K has the closed half-line property, then so has K^* .

Proof. The fact that K^* is closed is immediate, which settles item (1). Next we prove item (2). In dimension one a density point is an accumulation point and the set of all isolated points has measure zero. In general, for any one-dimensional linear subspace $\mathcal{L} \subseteq \mathbb{R}^n$, consider

$$K_{\mathcal{L}}^* := \{a \in K \mid a \text{ is accumulation point of } K \cap (x + \mathcal{L})\}.$$

By the Fubini theorem the set $K \setminus K_{\mathcal{L}}^*$ has measure zero. We use this result for a countable collection $\{\mathcal{L}_j\}_{j \in \mathbb{N}}$ of lines, which are dense in $\mathbb{P}^{n-1}(\mathbb{R})$. Indeed, then we have that $\bigcap_{j \in \mathbb{N}} K_{\mathcal{L}_j}^* \subseteq K^*$, which proves the result.

Finally, we turn to the closed half-line property of item (3). For any $p \in K^*$ and $s \geq 1$ we show that also $sp \in K^*$. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^∞ -function with $F|_K \equiv 0$, then we have to show that $j^\infty(F)(sp) = 0$. Defining $F_s(x) = F(sx)$, we observe that $j^\infty(F_s)(p) = 0$ if and only if $j^\infty(F)(sp) = 0$. By the closed half-line property of K it follows that $F_s|_K \equiv 0$ and hence $j^\infty(F_s)(p) = 0$. Therefore, $j^\infty(F)(sp) = 0$ and $sp \in K^*$. \square

We apply this notion for $K = D_{\tau, \gamma}(\mathbb{R}^n)$, thereby introducing the closed set

$$D_{\tau, \gamma}^*(\mathbb{R}^n) \subseteq D_{\tau, \gamma}(\mathbb{R}^n)$$

of (τ, γ) -Diophantine density points. Lemma 1 ensures that $D_{\tau, \gamma}(\mathbb{R}^n) \setminus D_{\tau, \gamma}^*(\mathbb{R}^n)$ has Lebesgue measure zero and inherits the closed half-line property from $D_{\tau, \gamma}(\mathbb{R}^n)$. In the same way we introduce sets $D_{\tau, \gamma}^*(A_\gamma)$, etc., with similar properties (compare with §1).

Remark. We note that the above definition of density does not coincide with that of Lebesgue density points. A general problem is to characterize $D_{\tau, \gamma}^*(\mathbb{R}^n) \subseteq D_{\tau, \gamma}(\mathbb{R}^n)$.

2.2. Formulation of the unicity theorem. Let the integrable system X , see (3), be given on $\mathbb{T}^n \times A$. We shall use C^∞ -neighbourhoods of X in the space of Hamiltonian vector fields, to be denoted by $\mathcal{U}_1, \mathcal{U}_2, \dots$. Similarly, we shall use C^∞ -neighbourhoods of the identity map $\text{Id}_{\mathbb{T}^n \times A}$ to be denoted by $\mathcal{V}_1, \mathcal{V}_2, \dots$. The KAM Theorem 1 can be rephrased as follows. If \mathcal{V}_1 is a neighbourhood of $\text{Id}_{\mathbb{T}^n \times A}$, then there exists a neighbourhood \mathcal{U}_1 of X such that for all $\tilde{X} \in \mathcal{U}_1$ there exists $\Phi \in \mathcal{V}_1$ conjugating X and \tilde{X} when restricting to $\mathbb{T}^n \times D_{\tau, \gamma}(A_\gamma)$. Our present aim is to prove the following statement.

THEOREM 2. (Unicity of KAM tori) *In the set-up of the KAM Theorem 1 and under the same conditions, there exist C^∞ -neighbourhoods $\mathcal{U}_2, \mathcal{V}_2$ such that the following holds. For all $\tilde{X} \in \mathcal{U}_2$ there exists $\Phi \in \mathcal{V}_2$ that:*

- (1) restricted to $\mathbb{T}^n \times D_{\tau,\gamma}(A_\gamma)$, conjugates X and \tilde{X} ;
- (2) moreover, the map Φ , after restriction to $\mathbb{T}^n \times D_{\tau,\gamma}^*(A_\gamma)$ is unique up to torus translations.

Remark. It is possible to generalize the present approach to cases where the phase space has a larger dimension while also introducing extra symmetry, which is preserved under perturbation. However, it is more difficult to deal with the KAM theory of lower dimensional, isotropic tori in Hamiltonian systems. In future work we shall come back to this, where also transitions of the corresponding torus bundles will be considered, both in the integrable and in the nearly-integrable setting. As an example think of the Hamiltonian Hopf bifurcation and higher dimensional variations of this [4, 3, 16]. More general information on this ‘quasi-periodic’ bifurcation theory can be found in [5–7, 9, 12].

2.3. *Set-up of a proof of the unicity Theorem 2.* Instead of Theorem 2 we shall prove the following.

LEMMA 2. *In the set-up of Theorem 2, there exist C^∞ -neighbourhoods $\mathcal{U}_3, \mathcal{V}_3$ such that if $\tilde{X} \in \mathcal{U}_3$ and if $\tilde{X}|_{\mathbb{T}^n \times D_{\tau,\gamma}(A_\gamma)} = X|_{\mathbb{T}^n \times D_{\tau,\gamma}(A_\gamma)}$, then, if $\Phi \in \mathcal{V}_3$ is a conjugacy between $X|_{\mathbb{T}^n \times D_{\tau,\gamma}(A_\gamma)}$ and $\tilde{X}|_{\mathbb{T}^n \times D_{\tau,\gamma}(A_\gamma)}$, one has that*

$$\Phi|_{T^n \times D_{\tau,\gamma}^*(A_\gamma)} \equiv \text{Id}_{\mathbb{T}^n \times D_{\tau,\gamma}^*(A_\gamma)}$$

modulo torus translations.

Lemma 2 implies Theorem 2 as follows. Given the neighbourhoods $\mathcal{U}_3, \mathcal{V}_3$, we choose $\mathcal{U}_2, \mathcal{V}_2$ in such a way that:

- for any $\Phi \in \mathcal{V}_2$ and $\tilde{X} \in \mathcal{U}_2$ we have that $\Phi_*^{-1} \tilde{X} \in \mathcal{U}_3$;
- for any $\Phi, \Psi \in \mathcal{V}_2$ we have that $\Psi^{-1} \circ \Phi \in \mathcal{V}_3$.

Remark. Note that the neighbourhoods $\mathcal{U}_j, \mathcal{V}_j, j = 1, 2, 3$, in the above argument are going to depend on τ and γ . Moreover, by compactness of $D_{\tau,\gamma}(A_\gamma)$, all constants in the estimates can be chosen uniform in $\omega \in D_{\tau,\gamma}^*(A_\gamma)$.

We set up the proof of Lemma 2. To this end we fix a frequency vector $\omega \in D_{\tau,\gamma}^*(A_\gamma) \subseteq D_{\tau,\gamma}(A_\gamma)$, so as a (τ, γ) -Diophantine density point. By a change of coordinates X gets the form

$$X = \sum_{j=1}^n (\omega_j + a_j) \frac{\partial}{\partial \alpha_j},$$

where the interest is with the n -torus $\mathbb{T}_0 := \mathbb{T}^n \times \{0\}$. By the density assumption we have

$$j^\infty(X - \tilde{X})|_{\mathbb{T}_0} \equiv 0.$$

Assume that there exists an \tilde{X} -invariant n -torus \mathbb{T} , conjugate to \mathbb{T}_0 . Denoting the conjugacy by ϕ , note that ϕ determines functions f and g such that

$$\phi(\alpha, 0) = (\alpha + f(\alpha), g(\alpha)).$$

Given \mathbb{T} , the conjugacy is unique up to translations, compare with the remark at the beginning of §2. The correspondence between (f, g) and ϕ therefore is unique when

we fix the average of f at $f_0 = 0$. In that case we also write $\phi = \phi_{f,g}$. We shall derive a contradiction from the assumption that f and g are small.

We first write

$$\bar{X}(\alpha, a) = (\omega + a + F) \frac{\partial}{\partial \alpha} + G \frac{\partial}{\partial a},$$

where for the infinite jets we have

$$j^\infty(F)|_{\mathbb{T}_0} \equiv 0 \equiv j^\infty(G)|_{\mathbb{T}_0}, \quad (4)$$

since $\omega \in D_{\tau, \gamma}^*(A_\gamma)$. The condition that $\Phi_{f,g}$ is a conjugacy now is expressed in the variable α and a as follows:

$$g(\alpha) + F(\alpha + f(\alpha), g(\alpha)) = \sum_{j=1}^n \omega_j \frac{\partial f}{\partial \alpha_j}(\alpha), \quad (5)$$

$$G(\alpha + f(\alpha), g(\alpha)) = \sum_{j=1}^n \omega_j \frac{\partial g}{\partial \alpha_j}(\alpha). \quad (6)$$

Next taking

$$\varepsilon := \|g\|_0 = \max_{\alpha} |g(\alpha)|, \quad (7)$$

we aim to show that for small positive ε a contradiction arises. Since we only consider conjugacies that are C^∞ -close to the identity map, we may assume that $\|(f, g)\|_m \leq 1$. Then, using (4), we shall obtain an appropriate flatness estimate

$$\|F(\alpha + f(\alpha), g(\alpha)), G(\alpha + f(\alpha), g(\alpha))\|_m \leq C_1 \varepsilon^2, \quad (8)$$

as ε is sufficiently small; this is called flat of order two in ε . For a proof see §2.4.

The argument to obtain a contradiction for small, positive ε roughly runs as follows. Let us write $g = g_0 + \tilde{g}$, where g_0 denotes the \mathbb{T}^n -average, which is constant, and \tilde{g} the part that varies with α . By (8) the left-hand side of (6) is flat of order two in ε . Therefore, also the right-hand side is flat of order two in ε , implying that in particular

$$\|\tilde{g}\|_0 = O(\varepsilon^2). \quad (9)$$

The estimate (9) will be shown in §2.4, using the fact that ω is Diophantine.

Using this order two flatness of \tilde{g} and the assumption (7), it follows that on the one hand

$$O(\varepsilon^2) = \|\tilde{g}\|_0 \geq \|g_0\| - \|g\|_0 = \|g_0\| - \varepsilon,$$

and hence that

$$\|g_0\| = O_S(\varepsilon), \quad (10)$$

meaning that $|g_0|$ has the sharp (or exact) order ε as $\varepsilon \rightarrow 0$.

On the other hand, consider equation (5), taking the \mathbb{T}^n -average of both sides. It is easily seen that the right-hand side has average zero, since it contains only derivatives of a smooth real function on \mathbb{T}^n . Therefore, also the left-hand side must have average zero, which means that

$$g_0 + \text{av}_{\mathbb{T}^n}[F(\alpha + f(\alpha), g(\alpha))] = 0. \quad (11)$$

From (8) we know that

$$\text{av}_{\mathbb{T}^n}[F(\alpha + f(\alpha), g(\alpha))] = O(\varepsilon^2), \quad (12)$$

which, together with (11), implies that

$$|g_0| = O(\varepsilon^2). \quad (13)$$

The estimate (13) clearly contradicts the sharp order ε -estimate (10).

In the next section, we shall provide precise versions of the estimates (8) and (9). For the moment, by taking these for granted we may conclude that this contradiction proves Lemma 2 and thereby the unicity Theorem 2.

2.4. Estimates for Lemma 2. In this section, we shall introduce several constants that determine the neighborhoods \mathcal{U}_3 and \mathcal{V}_3 . As said before, the corresponding estimates hold uniformly for all $\omega \in D_{\tau, \gamma}^*(A_\gamma)$.

Recall that (4) expresses that

$$j^\infty(X - \bar{X})_{\mathbb{T}^n \times D_{\tau, \gamma}^*(A_\gamma)} \equiv 0,$$

from which we obtain a Taylor formula

$$|j^\ell(X - \bar{X})(\alpha, a)| \leq C_{m, \ell}^{(0)} \varrho^m, \quad (14)$$

where ϱ is the distance of (α, a) to $\mathbb{T}^n \times D_{\tau, \gamma}^*(A_\gamma)$.

The functions $f, g, F(\cdot, a)$ and $G(\cdot, a)$ can all be expanded in Fourier series, yielding the respective corresponding Fourier coefficients $f_k, g_k, F_k(a)$ and $G_k(a)$, for $k \in \mathbb{Z}^n$. We will need the following. For any formal Fourier series

$$h(\alpha) = \sum_{k \in \mathbb{Z}^n} h_k e^{i\langle \alpha, k \rangle}$$

we introduce the following familiar norms in terms of the Fourier coefficients:

$$\begin{aligned} \|h\|_\infty &= \max_k |h_k|; \\ \|h\|_2 &= \sqrt{\sum_k |h_k|^2}; \\ \|h\|_1 &= \sum_k |h_k|. \end{aligned}$$

For any (continuous) function $h : \mathbb{T}^n \rightarrow \mathbb{R}$ we recall that

$$\|h\|_\infty \leq \|h\|_2 \leq \|h\|_0 \leq \|h\|_1. \quad (15)$$

Since we did not require the Fourier series to converge, some of these norms may be infinite.

LEMMA 3. *Let $h : \mathbb{T}^n \rightarrow \mathbb{R}$ be of class C^m , with variable part \tilde{h} .*

(1) *Then there exists a positive constant $C_{m, n}^{(1)}$ such that for all $k \in \mathbb{Z}^n \setminus \{0\}$*

$$|k|^m |h_k| \leq C_{m, n}^{(1)} \|\tilde{h}\|_m.$$

- (2) Moreover, for $m \geq n + 1$ there exists a positive constant $C_{m,n}^{(2)}$ such that

$$\|\tilde{h}\|_0 \leq C_{m,n}^{(2)} \max |k|^m |h_k|.$$

- (3) Assume that h satisfies the differential equation

$$\sum_{j=1}^n \omega_j \frac{\partial h(\alpha)}{\partial \alpha_j} = H(\alpha),$$

with a given $H : \mathbb{T}^n \rightarrow \mathbb{R}$ of class C^m with \mathbb{T}^n -average 0 and where ω is (τ, γ) -Diophantine. Then

$$|h_k| \leq \frac{|k|^\tau}{\gamma} |H_k|$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$. Moreover, for $m \geq n + 1 + \tau$ one has that $h \in C^0(\mathbb{T}^n, \mathbb{R})$ with

$$\|\tilde{h}\|_0 \leq C_{m,n,\tau}^{(3)} \frac{1}{\gamma} \|H\|_m,$$

for a positive constant $C_{m,n,\tau}^{(3)}$.

We note that the estimates in Lemma 3 are rather crude, but yet exactly appropriate for the present purposes.

Proof. We recall that $k = (k_1, k_2, \dots, k_n)$, where $|k| = |k_1| + \dots + |k_n|$. Also

$$h_k = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} e^{-i\langle k, \alpha \rangle} h(\alpha) d\alpha.$$

The first inequality is the familiar Paley–Wiener decay of the Fourier series of C^m -functions. For completeness we include a proof. By partial integration it follows that

$$h_k = \frac{1}{(2\pi)^n} \frac{1}{(ik_1)^{p_1} \dots (ik_n)^{p_n}} \int_{\mathbb{T}^n} e^{-i\langle k, \alpha \rangle} \frac{\partial^p h}{\partial \alpha_1^{p_1} \dots \partial \alpha_n^{p_n}}(\alpha) d\alpha, \quad (16)$$

where $p_1 + \dots + p_n \leq m$. First for $p = (p_1, \dots, p_n)$, $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ we introduce the multi-index notation $k^p = k_1^{p_1} \dots k_n^{p_n}$. Also, we write $\mathcal{P}_m = \{p \in \mathbb{Z}^n \mid \sum_{j=1}^n p_j = m\}$. A consideration on positive definite, homogeneous polynomials next reveals that

$$\sum_{p \in \mathcal{P}_m} k^p \sim |k|^m. \quad (17)$$

From (16) it now follows that

$$k^p |h_k| \leq C \|\tilde{h}\|_m$$

and hence

$$\sum_{p \in \mathcal{P}_m} k^p |h_k| \leq C' \|\tilde{h}\|_m,$$

which by (17) directly gives the desired estimate.

For item (2) we introduce $D_m(h) = \max |k|^m |h_k|$. It follows that $|h_k| \leq D_m(h)/|k|^m$ for all $k \neq 0$. Then, for the variable part \tilde{h} of h we have

$$\|\tilde{h}\|_0 \leq \|\tilde{h}\|_1 = \sum_{k \neq 0} |h_k| \leq D_m(h) \sum_{k \neq 0} \frac{1}{|k|^m},$$

where we have used (15) and where the latter sum is bounded for $m \geq n + 1$. Hence, for $C_{m,n}^{(2)} = \sum_{k \neq 0} 1/|k|^m$ we have

$$\|\tilde{h}\|_0 \leq C_{m,n}^{(2)} D_m(h),$$

as was to be shown.

For item (3), comparing terms in the Fourier series, we find that

$$i \langle \omega, k \rangle h_k = H_k,$$

for $k \neq 0$, which by the Diophantine condition (2) gives the desired estimate on the Fourier coefficients h_k . Next, if $m \geq n + 1 + \tau$, we obtain

$$\begin{aligned} \|\tilde{h}\|_0 &\leq \|\tilde{h}\|_1 \leq \sum_{k \neq 0} \frac{|k|^\tau}{\gamma} |H_k| \\ &\leq \sum_{k \neq 0} \frac{|k|^\tau}{\gamma} |k|^{-m} n^m \|H\|_m \leq \frac{C}{\gamma} \sum_{\ell \neq 0} \ell^{\tau-m} \|H\|_m \ell^{n-1} \\ &\leq C \frac{\|H\|_m}{\gamma} \sum_{\ell \neq 0} \ell^{n+\tau-m-1} \leq C' \frac{\|H\|_m}{\gamma}, \end{aligned}$$

where we use the fact that $n + \tau - m - 1 \leq -2$, which gives convergence of the sum and yields the final estimate. Compare with the items (1) and (2). \square

Remark. Note that if $m \geq n + p + \tau$ with $p \geq 1$, it follows that $h \in C^{p-1}$, where similarly

$$\|\tilde{h}\|_{p-1} \leq C'' \frac{\|H\|_m}{\gamma},$$

where we now use the fact that $n + \tau - m - 1 \leq -1 - p$.

We also need the following estimate on the m -jet of a composed map.

LEMMA 4. *Consider the m -jet of $U \circ V$, where all derivatives of V up to order m are bounded by 1. Then,*

$$\|U \circ V\|_m \leq C_m^{(4)} \|U\|_m,$$

with a positive constant $C_m^{(4)}$.

Proof. By the chain rule, the m -jet of a composition $U \circ V$ is linear in the m -jet of U and polynomial in the m -jet of V . We restricted to a compact set of m -jets for V , by taking all derivatives up to order m less than or equal to 1. It then follows that $\|U \circ V\|_m \leq C_m^{(4)} \|U\|_m$, for a positive constant $C_m^{(4)}$. \square

The following proposition sums up the above ingredients and provides the uniform estimates (8) and (9), used in the argument concluding §2.3.

PROPOSITION. *Let $\ell \geq 2$ and $m \geq n + 1 + \tau$. Assume that $\|(f, g)\|_m \leq 1$ and that $\|g\|_0 = \varepsilon$, then there exists $\varepsilon_0 > 0$, independent of $\omega \in D_{\tau,\gamma}^*(A_\gamma)$, such that for all*

$0 < \varepsilon \leq \varepsilon_0$ we have the following. Denoting by \tilde{g} the variable part of g and abbreviating $\hat{F}(\alpha) = F(\alpha + f(\alpha), g(\alpha))$ and $\hat{G}(\alpha) = G(\alpha + f(\alpha), g(\alpha))$, we have

$$\|\hat{F}\|_0 \leq C_0^{(4)} C_{0,\ell}^{(0)} \varepsilon^\ell, \quad (18)$$

$$\|\hat{G}\|_m \leq C_m^{(4)} C_{m,\ell}^{(0)} \varepsilon^\ell, \quad (19)$$

$$\|\tilde{g}\|_0 \leq C_{m,n,\tau}^{(3)} C_m^{(4)} \frac{\varepsilon^\ell}{\gamma}. \quad (20)$$

Proof. Note that by (6) the \mathbb{T}^n -average of \hat{G} vanishes: indeed, the right-hand side only contains derivatives of a function on \mathbb{T}^n . First, by Lemma 4 and the Taylor estimate (14)

$$\|\hat{G}\|_m \leq C_m^{(4)} \|G\|_m \leq C_m^{(4)} C_{m,\ell}^{(0)} \varepsilon^\ell,$$

which proves (19). Similarly, taking $m = 0$, we obtain the inequality (18) for \hat{F} . Observe that the first estimate of Lemma 3 yields, in combination with (19), that

$$|\hat{G}_k| \leq |k|^{-m} C_{m,n}^{(1)} \|\hat{G}\|_m \leq C_{m,\ell}^{(0)} C_{m,n}^{(1)} C_m^{(4)} |k|^{-m} \varepsilon^\ell.$$

Next, to prove (20) we observe that, by (6), we have

$$\sum_{j=1}^n \omega_j \frac{\partial g(\alpha)}{\partial \alpha_j} = \hat{G}(\alpha),$$

to which we apply item (3) of Lemma 3, using the fact that $\omega \in D_{\tau,\gamma}(A_\gamma)$. It follows that for all $k \neq 0$

$$|g_k| \leq |\hat{G}_k| \frac{|k|^\tau}{\gamma} \leq C_{m,\ell}^{(0)} C_{m,n}^{(1)} C_m^{(4)} \frac{|k|^{\tau-m}}{\gamma} \varepsilon^\ell$$

and also (20) directly follows. Compare with the proof of Lemma 3. \square

We return to the details of the argument at the end of §2.3, observing that the above proposition provides the necessary estimates (8), (9) and (12). In fact, in the proposition we fix $\ell = 2$ and $m = n + \tau + 1$ (with $\tau \in \mathbb{N}$, e.g. taking $\tau = n$). In this way (18) and (19) are obtained as the appropriate form of (8), also directly implying (12). The estimate (9) follows by (20). We note that all estimates are uniform in $\omega \in D_{\tau,\gamma}^*(A_\gamma)$.

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